The generation of edge waves by moving pressure distributions

By H. P. GREENSPAN
Harvard University, Cambridge, Massachusetts

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SUMMARY

An analytical study is made of the resurgent wave motion induced by pressure distributions moving parallel to a straight coast line. The resurgence is shown to consist of an infinite number of edge wave modes; expressions for these modes are given and the wavelengths, frequencies and amplitudes are shown to be in agreement with experimental results. The effects of a Gaussian pressure distribution are analysed. For large-scale disturbances off the east coast of the United States only the fundamental mode is excited. Conditions, such as storm size, speed and distance from shore, for maximum induced wave amplitudes are derived.

1. INTRODUCTION

It has been observed that hurricanes and other similar phenomena travelling approximately parallel to a neighbouring coast line sometimes induce a resurgent wave motion. These waves are characterized by the facts that they propagate along the shore line, are essentially confined within a distance of one wavelength from the coast (hence the name 'edge waves'), have wavelengths measured in miles, amplitudes measured in feet, and periods measured in hours.

Munk, Snodgrass & Carrier (1956) conjectured that this resurgence was primarily due to the deviation of the storm pressure distribution from normal atmospheric pressure, and was neither wind-generated nor a consequence of other effects. Using a simple mathematical model for the pressure deviation, with the assumption of quasi-steady state conditions (that the disturbance has been in existence and in motion for an infinite time interval), they obtained results in agreement with experimental evidence. Computed periods, and amplitudes agreed with known values. Their original conjecture concerning the cause of the resurgence was thereby substantiated.

In this paper, the transient problem (i.e. with the disturbance originating at a finite time) is considered and applied to a more precise physical model.

2. THE FUNDAMENTAL EQUATION AND ITS SOLUTION

In order to motivate the choice of an appropriate model on which to base our calculations, we anticipate (as indicated in the introduction) that the hurricane generated surface waves will display an amplitude η of the order of 3 feet, a wavelength λ of about 200 miles, and a period of about 6 hours. The extremely long wavelengths and small wave heights suggest that the linear shallow water theory may be applied. In order that this theory may provide a valid approximation, the wavelength of a progressive wave propagation must be large compared to the ocean depth, the ocean depth must be large compared to the wave height, and the gradients of the wave height and depth must be small compared to unity. Simple estimates of the orders of magnitude involved show that in the phenomena in question these conditions are satisfied.

The linearized theory essentially eliminates the depth variable z from the exact analysis by averaging the velocity components over the depth. The pressure is then hydrostatic (see Lamb 1945 for a more complete analysis).

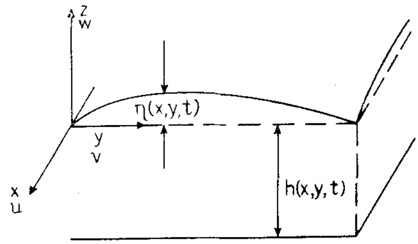

Figure 1. Fluid with a fixed boundary and a free surface.

With the meaning of the symbols shown in figure 1, and with uniform density ρ, the conservation equations of mass and momentum are

$$\frac{\partial\eta(x,y,t)}{\partial t} = -\frac{\partial(uh)}{\partial x} - \frac{\partial(vh)}{\partial y}; \tag{1}$$

$$\frac{\partial u(x,y)}{\partial t} = -\frac{1}{\rho}\frac{\partial P(x,y,z,t)}{\partial x}; \qquad \frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial P(x,y,z,t)}{\partial y}, \tag{2}$$

where $P(x,y,z,t) \doteqdot P(x,y,0,t)+g\rho(\eta-z)$, and $P(x,y,0,t)$ is the applied surface pressure. We may eliminate u and v to obtain

$$\nabla.(h\nabla\eta)-\frac{1}{g}\frac{\partial^2\eta}{\partial t^2} = -\frac{1}{\rho g}(h\nabla^2P+\nabla h.\nabla P). \tag{3}$$

Here and in the following, $P(x,y,0,t)$ is replaced by P or $P(x,y,t)$. We can also define a potential function by the relations $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$, and eliminate η to obtain an equation for ϕ:

$$\nabla.(h\nabla\phi)-\frac{1}{g}\frac{\partial^2\phi}{\partial t^2} = \frac{1}{\rho g}\frac{\partial P}{\partial t}, \tag{4}$$

the wave height being

$$\eta = -\frac{P}{\rho g}-\frac{1}{g}\frac{\partial\phi}{\partial t}. \tag{5}$$

It remains to specify the ocean depth h, and to linearize the topography. We consider only a straight coast line, and take the ocean depth to be a linear function of distance from shore y, i.e. $h = \alpha y$. For the constant α, we choose the average slope of the ocean depth over the first 120 miles. Munk, Snodgrass & Carrier (1956) have shown that since the wave motion under investigation is sensibly zero farther than $\lambda/2\pi$ from shore, it is essentially independent of the manner in which the ocean depth varies at these large distances. Since the phenomenon in which we are primarily interested is essentially confined to the continental shelf, the approximation of constant slope is adequate. The total linearization of the actual topography is then a straight coast line and a constant slope depth (figure 2).

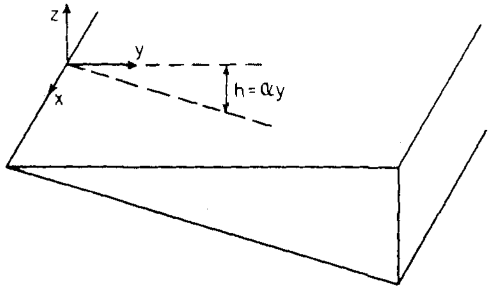

Figure 2. Linearized ocean depth and coast line topography.

With $h = \alpha y$, equation (3) becomes

$$y\nabla^2\eta + \frac{\partial\eta}{\partial t} - \frac{1}{\alpha g}\frac{\partial^2\eta}{\partial t^2} = -q_1(x,y,t) = -\frac{1}{\rho g}\left(\frac{\partial P}{\partial y} + y\nabla^2 P\right), \tag{6}$$

and equation (4) becomes

$$y\nabla^2\phi + \frac{\partial\phi}{\partial t} - \frac{1}{\alpha g}\frac{\partial^2\phi}{\partial t^2} = -q_2(x,y,t) = \frac{1}{\rho g\alpha}\frac{\partial P}{\partial t}. \tag{7}$$

Given that $P(x,y,t) = 0$ for $t < 0$, the correct boundary conditions are: $\eta(x,y,0) = 0$, $\partial\eta(x,y,0)/\partial t = 0$, $\eta(x,y,t) \to 0$ as $|x| \to \infty$ for all positive y, and also as $y \to \infty$ for all x (the phenomenon is confined to the shelf), and finally that the wave height is finite at the shore line. This finiteness condition is the most liberal physical requirement one can apply in the coastal region where the linearized differential equations are not a valid description.

Since equations (6) and (7) differ only in the form of the forcing function, we need only solve the former; the solution to the latter is obtained by replacing η by ϕ and $q_1(x,y,t)$ by $q_2(x,y,t)$. We can now take Fourier and Laplace transforms of equation (6). Defining

$$\left.\begin{aligned} f(k,y,s) &= \int_{-\infty}^{\infty} e^{ikx}\,dx \int_0^{\infty} e^{-st}\,\eta(x,y,t)\,dt \\ Q_j(k,y,s) &= \int_{-\infty}^{\infty} e^{ikx}\,dx \int_0^{\infty} e^{-st}\,q_j(x,y,t)\,dt, \end{aligned}\right\} \tag{8}$$

where $j = 1, 2$, we obtain

$$y\frac{d^2f}{dy^2} + \frac{df}{dy} - \left(\frac{s^2}{\alpha g} + k^2 y\right)f = -Q_1(k,y,s). \tag{9}$$

The boundary conditions become $|f(k,0,s)| < \infty$ and $f(k,y,s) \to 0$ as $y \to \infty$. Anticipating difficulties with the inverse Fourier transform, we shall solve, instead of equation (9),

$$y\frac{d^2f}{dy^2}+\frac{df}{dy}-\left(\frac{s^2}{\alpha g}+k_*^2 y\right)f = -Q_1(k,y,s), \tag{10}$$

where

$$k_* = (k^2+\epsilon^2)^{1/2}, \tag{11}$$

subject to the same boundary conditions. After taking the inverse Laplace and Fourier transforms of the solution of equation (10), we shall let ϵ tend to zero to obtain the solution of equation (6). If it is convenient and proper, we can also perform this limiting process at some intermediate step. The function k_* is not single-valued in the complex k plane and we restrict its possible values by two branch lines originating at $\pm i\epsilon$ and extending to $\pm i\infty$ respectively. The reasons for this particular choice will become evident shortly. We choose the Riemann sheet for which $k_* \geqslant 0$ on the entire real axis.

A few simple substitutions will enable us to rewrite equation (10) in a more convenient form. Let

$$\left.\begin{aligned} f(k,y,s) &= \exp(-k_* y)p(k,y,s),\\ u &= 2k_* y,\\ p_1(k,u,s) &= p(k,\tfrac{1}{2}u/k_*,s), \end{aligned}\right\} \tag{12}$$

and equation (10) becomes

$$up_1''+(1-u)p_1'-\left(\frac{1}{2}+\frac{s^2}{2\alpha g k_*}\right)p_1 = -\frac{\exp(\frac{1}{2}u)}{2k_*}Q_1(k,\tfrac{1}{2}u/k_*,s). \tag{13}$$

The boundary conditions become $\exp(-\frac{1}{2}u)p_1(u) \to 0$ as $u \to \infty$ and $|p_1(k,0,s)| < \infty$*. To solve (13), we expand $p_1(k,y,s)$ and

$$-\frac{\exp(\frac{1}{2}u)}{2k_*}Q_1(k,\tfrac{1}{2}u/k_*,s)$$

in terms of Laguerre polynomials (see Courant & Hilbert 1953), and then use the differential equation to determine the unknown coefficients. The expansions are

$$p_1(u) = p_1(k,u,s) = \sum_{n=0}^{\infty} A_n L_n(u), \tag{14}$$

$$-\frac{\exp(\frac{1}{2}u)}{2k_*}Q_1(k,\tfrac{1}{2}u/k_*,s) = \sum_{n=0}^{\infty} B_n L_n(u), \tag{15}$$

where

$$L_n(u) = e^u\left(\frac{d^n}{du^n}\right)(u^n e^{-u}), \tag{16}$$

* We can consider k and s to be real numbers until it is necessary to invert the Fourier and Laplace transforms.

and
$$\left.\begin{aligned} A_n &= \frac{1}{(n!)^2}\int_0^\infty e^{-u}p_1(u)L_n(u)\,du, \\ B_n &= -\frac{1}{(n!)^2}\int_0^\infty \frac{\exp(-\frac{1}{2}u)}{2k_*}\,Q_1(k,\tfrac{1}{2}u/k_*,s)L_n(u)\,du. \end{aligned}\right\} \tag{17}$$

Substituting these expansions into equation (13) and using the fact that $L_n(u)$ satisfies Laguerre's differential equation
$$uL_n''(u)+(1-u)L_n'(u)+nL_n(u) = 0, \tag{18}$$
we obtain
$$\sum_0^\infty A_n\left(-n-\tfrac{1}{2}-\frac{s^2}{2\alpha gk_*}\right)L_n(u) = \sum_0^\infty B_n\,L_n(u). \tag{19}$$
We use the orthogonality relation
$$\int_0^\infty e^{-u}L_n(u)L_m(u)\,du = (n!)^2\delta_{mn} \tag{20}$$
to equate coefficients, and hence determine that
$$A_n = \frac{-B_n}{(n+\frac{1}{2}+s^2/2\alpha gk_*)}. \tag{21}$$
This implies that
$$p_1(u) = p_1(k,u,s) = -\sum_{n=0}^\infty \frac{B_n\,L_n(u)}{(n+\frac{1}{2}+s^2/2\alpha gk_*)} \tag{22}$$
($p_1(u)$ satisfies the boundary conditions, since each $L_n(u)$ does) or
$$f(k,y,s) = \exp(-k_*y)\sum_{n=0}^\infty \frac{\int_0^\infty L_n(u)\exp(-\frac{1}{2}u)Q_1(k,\frac{1}{2}u/k_*,s)\,du}{(n+\frac{1}{2}+s^2/2\alpha gk_*)2k_*(n!)^2}\,L_n(2k_*y). \tag{23}$$
The wave height is then obtained by inverting the Fourier and Laplace transforms; i.e.
$$\eta(x,y,t) = \frac{1}{4\pi^2 i}\int_{-\infty}^\infty e^{-ikx}\,dk\int_{-i\infty+\epsilon_0}^{i\infty+\epsilon_0} e^{st}f(k,2,s)\,ds. \tag{24}$$
The path of integration in the k plane for the inverse Fourier transform must lie within a strip containing the real axis. The function k_* has been defined with this in mind; i.e. the inversion contour can pass between the two branch points $\pm i\epsilon$ without crossing a branch line.

There are other ways to solve equation (13); methods involving the solutions of the homogeneous equation (i.e. confluent hypergeometric functions) lead to extremely difficult integrations. A Green's function technique for solving (6), although of little practical value, reduces to an interesting eigenvalue problem, one with both a discrete and a continuous spectrum; this will be the subject of a future paper. The advantage of the method used lies in the fact that we shall be able to consider separately the effects of a pressure distribution on the individual edge wave modes, easily eliminating extraneous parts of the solution. The wave motion can be determined with little difficulty.

3. Application to the model of Munk, Snodgrass & Carrier

Let us now consider this model, adjusted so that it arises at some finite time. The surface pressure is

$$P(x,y,t) = \frac{P_0 a(y+a)}{(x-Ut)^2+(y+a)^2} H(t), \tag{25}$$

where $H(t)$ is the Heaviside function, U is the velocity at which the disturbance moves parallel to the coast, and a is the half-pressure radius. The resemblance to actual hurricanes is poor, but the Fourier transform is simple. We shall solve for the wave height η directly. The forcing function is, from equation (6),

$$q_1(x,y,t) = \frac{P_0 a}{\rho g} \frac{(x+Ut)^2-(y+a)^2}{[(x-Ut)^2+(y+a)^2]^2},$$

and its transform is

$$Q_1(k, \tfrac{1}{2}u/k_*, s)\frac{\exp(\frac{1}{2}u)}{2k_*} = -\frac{\pi P_0 a}{\rho g}\frac{k_* \exp(-ak_*)}{(s-ikU)2k_*},$$

which is constant, so that

$$B_0 = \frac{\pi P_0 a}{\rho g}\frac{k_* \exp(-ak_*)}{2k_*(s-ikU)}, \quad \text{and } B_n = 0 \text{ for } n \neq 0.$$

Thus the expansion of equation (22) reduces to one in which only the 'first' Laguerre polynomial, $L_0(u) = 1$, appears. This implies, as we shall see, that this pressure distribution can excite only the fundamental edge wave mode. The Fourier Laplace transform of η is

$$f(k,y,s) = \frac{\gamma k_* \exp\{-k_*(y+a)\}}{(s-ikU)(s^2+\alpha g k_*)}, \tag{26}$$

where

$$\gamma = -\frac{\alpha\pi P_0 a}{\rho}. \tag{27}$$

The wave height is determined by taking inverse Laplace and Fourier transforms as in equation (24). The function $f(k,y,s)$ is not meromorphic in k, but is in the variable s, having simple poles at $s = ikU, s = \pm i(\alpha g k_*)^{1/2}$. It is, therefore, much simpler to compute first the inverse Laplace transform. We set

$$r(k,y,t) = \frac{1}{2\pi i}\int_{-i\infty+\epsilon_0}^{i\infty+\epsilon_0} f(k,y,s)e^{st}\,ds,$$

and upon performing the integration we obtain for $t > 0$

$$r(k,y,t) = -\gamma k_* \exp\{-k_*(y+a)\}\left\{\frac{\exp(ikUt)}{k^2U^2-\alpha g k_*} + \right.$$
$$\left. + \frac{\exp\{i(\alpha g k_*)^{1/2}t\}}{2(\alpha g k_*)^{1/2}\{(\alpha g k_*)^{1/2}-kU\}} + \frac{\exp\{-i(\alpha g k_*)^{1/2}t\}}{2(\alpha g k_*)^{1/2}\{(\alpha g k_*)^{1/2}+kU\}}\right\}, \tag{28}$$

and $r(k,y,t) = 0$ for $t < 0$.

As a function of k, $r(k,y,t)$ has *no* poles on the real axis. The solution of the problem is then

$$\eta(x,y,t) = \frac{1}{2\pi}\int_\Gamma e^{-ikx}r(k,y,t)\,dk, \tag{29}$$

or

$$\eta(x,y,t) = -\frac{\gamma}{2\pi}(I+II+III), \tag{30}$$

where

$$\left.\begin{aligned} I &= \int_\Gamma f_I\,dk = \int_\Gamma \frac{\exp\{-k_*(y+a)\}\exp\{-ik(x-Ut)\}}{(k_*U^2-\alpha g)}\,dk, \\ II &= \int_\Gamma f_{II}\,dk = \int_\Gamma \frac{\exp[-i\{kx-(\alpha g k_*)^{1/2}t\}]k_*\exp\{-k_*(y+a)\}}{2(\alpha g k_*)^{1/2}\{(\alpha g k_*)^{1/2}-kU\}}\,dk, \\ III &= \int_\Gamma f_{III}\,dk = \int_\Gamma \frac{\exp[-i\{kx+(\alpha g k_*)^{1/2}t\}]k_*\exp\{-k_*(y+a)\}}{2(\alpha g k_*)^{1/2}\{(\alpha g k_*)^{1/2}+kU\}}\,dk, \end{aligned}\right\} \tag{31}$$

Although the original integrand has no poles, and the path Γ may be the real axis, each individual term does have one or more poles; Γ must therefore be suitably altered. We can easily verify that (30) is a solution of equation (6) by formally carrying out the differentiations.

Our task is now to evaluate and interpret these integrals. I is the easiest and we evaluate it first. This term represents the quasi-steady state solution derived by Munk, Snodgrass & Carrier (1956). (The terms due to a finite starting time are therefore the last two integrals.) The best contour to take is one that minimizes the effect in front of the pressure spot and produces a reaction behind it. Any contour will finally give a correct result for η provided we are consistent; however, we wish to give a definite physical meaning to I, independent of II and III. The poles of f_I are located at $k_*U^2-\alpha g = 0$, i.e.

$$\left.\begin{aligned} k_r &= k_0 = \frac{\alpha g}{U^2}, \\ k_l &= -k_0 = -\frac{\alpha g}{U^2}, \end{aligned}\right\} \tag{32}$$

for $\epsilon = 0$; k_r is located in the right half plane, and k_l in the left half plane. The best contour is one that goes under both of these singularities as in figure 3. For $x > Ut$, we may close the contour Γ in the lower half plane by the quadrants, Γ_1, Γ_1', the two sides of the branch line Γ_4, and the circular path about the branch point, as shown; for $x < Ut$ we may similarly close the contour in the upper half plane. The poles contribute values for $x < Ut$ only. The integrals about the branch lines are easily computed. The integrals over connecting quadrants tend to zero as $|k| \to \infty$, and the integrals round the small circles about branch points tend to zero as the radius $|k|$ tends to zero. The result is

$$\left.\begin{aligned} I &= \frac{2}{U^2}\operatorname{Re}\Psi(1,1,ik_0[|x-Ut|+i(y+a)]) \quad x < Ut,\\ I &= \frac{4\pi}{U^2}\exp\{-k_0(y+a)\}\sin k_0(x-Ut)+\\ &\quad+\frac{2}{U^2}\operatorname{Re}\Psi(1,1,ik_0[|x-Ut|+i(y+a)]) \quad x < Ut \end{aligned}\right\} \tag{33}$$

where Ψ is the confluent hypergeometric function of the second kind (see Erdelyi 1954). Asymptotically, for large values of $(x-Ut)^2+(y+a)^2$,

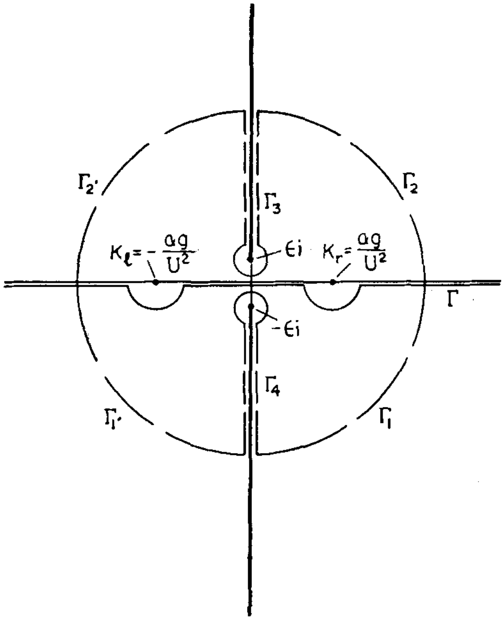

Figure 3. Fourier inversion contours for $X <$ or > 0.

Ψ is proportional to the surface pressure $P(x,y,t)$. The result consists of a resurgence for $x < Ut$, together with a motion restricted to the immediate neighbourhood of the pressure spot for which the wave height is the direct effect of the distribution on the ocean surface.

We now show that *II* and *III* essentially yield a cancelling wave, $-(4\pi/U^2)\exp\{-k_0(y+a)\}\sin k_0(x-Ut)$ for $x < \frac{1}{2}Ut$, implying that the resurgence is given asymptotically by

$$\eta \sim \frac{2\pi P_0 a k_0}{\rho g}\exp\{-k_0(y+a)\}\sin k_0(x-Ut) \quad \text{for } \tfrac{1}{2}Ut < x < Ut$$

$$\eta \sim 0 \text{ elsewhere,}$$

where k_0 is $\alpha g/U^2$ as in (32).

The integral to be evaluated is

$$II = \int_{\Gamma} \frac{\exp[-i\{kx-(\alpha g k_*)^{1/2}t\}-k_*(y+a)]}{2(\alpha g k_*)^{1/2}\{(\alpha g k_*)^{1/2}-kU\}} k_* \, dk.$$

To obtain results which are applicable when x and Ut are both large compared to $y+a$, we use a steepest descent procedure; it is noted that if

$$\exp\{-k_*(y+a)\}$$

were not present, we could integrate explicitly. The method of steepest descents is used to remove $\exp\{-k_*(y+a)\}$ from the integrand, and the remaining integral is then evaluated.

Let $x < 0$, $t = x/\beta$, then $II = \int_{\Gamma} \exp\{xf(k)\}\psi(k)\, dt,$

where

$$f(k) = -i\left(k-\frac{(\alpha g k_*)^{1/2}}{\beta}\right), \quad \text{and} \quad \psi(k) = \frac{\exp\{-k_*(y+a)\}k_*}{2(\alpha g k_*)^{1/2}\{(\alpha g k_*)^{1/2}-kU\}}.$$

We are interested in x and Ut large compared to $y+a$; accordingly $\exp\{-k_*(y+a)\}$ is included in the slowly-varying function $\psi(k)$. The saddle point, determined from the condition $f'(s_0) = 0$ is

$$s_0 = \frac{\alpha g}{4\beta^2}. \tag{34}$$

The path of steepest descent is (if $k = \xi + i\mu$, and $\epsilon = 0$) the parabola

$$\xi^2 - \left(\frac{\alpha g}{4\beta^2}\right)^2 = -\frac{\alpha g \mu}{2\beta^2}. \tag{35}$$

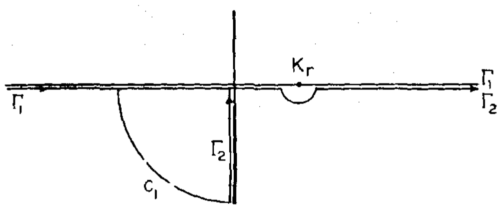

Figure 4. Contour deformation of the inversion path.

By considering the properties of the integrand f_{II}, and, specifically, the behaviour of the factor

$$\exp[-ikx+(\alpha g k_*)^{1/2}-k_*(y+a)]$$

as $|k|$ tends to infinity in the third quadrant, we can show that

$$\int_{\Gamma} f_{II}\, dk = \int_{\Gamma_2} f_{II}\, dk$$

(see figure 4). Thus far it has been essential to keep one branch line in the upper half plane, and the other in the lower half plane. The branch line from $i\epsilon$ to $i\infty$ can now be turned into the third quadrant (figure 5); and, for computing purposes, it will be taken to be the ray $\theta = \frac{3}{2}\pi$, where the angle θ has its usual meaning. We use the contour Γ_3 about the branch line and the paths Γ_4 and Γ_5, which are the arcs of a circle of radius R, to form a closed loop (in the limit as $R \to \infty$) with the path of steepest descent. Cauchy's theorem states that

$$\oint f_{II}\, dk = 2\pi i \sum \operatorname{res} f_{II}, \tag{36}$$

and thereby relates $\int_{\Gamma_2} f_{II}\, dk$ to the integral over the path of steepest descent $\int_{S.D.} f_{II}\, dk$ (the directions of integration are shown in figure 5). The integrals over the connecting arcs tend to zero as R tends to infinity, with the result that

$$\int_{\Gamma_2} f_{II}\, dk = \int_{S.D.} f_{II}\, dk + \int_{\Gamma_3} f_{II}\, dk + 2\pi i \sum \operatorname{res} f_{II}, \tag{37}$$

where the residues are evaluated only at those singularities contained within the closed contour. The pole of f_{II} ($k_r = k_0 = \alpha g/U^2$) is within the contour only when $s_0 > k_r$, i.e. for $\beta < \frac{1}{2}U$ or $x < \frac{1}{2}Ut$. For x large, the integral

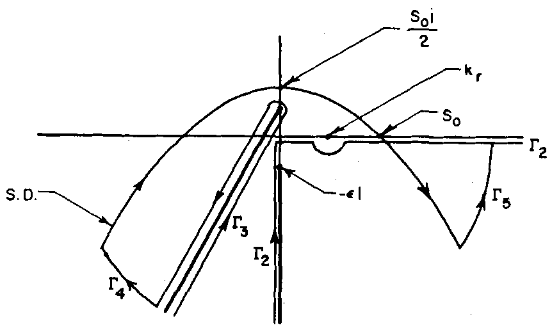

Figure 5. Conversion from the fundamental contour to the path of the steepest descent.

about the branch line is negligible compared with the integral along the path of steepest descent in the neighbourhood of the saddle point, and with the residue. (This is the advantage of the 'deformed branch line–steepest descent' method over other more direct estimations.) Hence we obtain

$$II \sim \int_{S.D.} f_{II}\, dk \tag{38}$$

$$-\frac{2\pi i}{U^2} \exp[-ik_0 x + i(\alpha g k_0)^{1/2} t - k_0(y+a)] H(\tfrac{1}{2}U - \beta),$$

where H is the Heaviside unit function. If the integral over the path of steepest descent were evaluated by conventional means as detailed by Morse & Feshbach (1953), the result would be

$$II \sim \left(\frac{\pi\alpha g}{x\beta^2}\right)^{1/2} \frac{1}{2\alpha g(1-U/2\beta)} \exp[-\tfrac{1}{4}\alpha g(y+a)/\beta^2 + \tfrac{1}{4}i\alpha g x/\beta^2 - \tfrac{1}{4}i\pi] -$$
$$-\frac{2\pi i}{U^2} \exp[-ik_0 x + i(\alpha g k_0)^{1/2} t - k_0(y+a)] H(\tfrac{1}{2}U - \beta). \tag{39}$$

In the neighbourhood of $s_0 = k_r$ ($\beta = \frac{1}{2}U$ or $x = \frac{1}{2}Ut$), this approximation fails, since the pole of the integrand is too near the saddle point, i.e. the 'slowly' varying function is the dominant factor. We can avoid this difficulty by slightly modifying the conventional technique. Instead of evaluating the *entire* integrand at the saddle point, we evaluate only part of it, i.e. $\exp[-k_*(y+a)]$. We obtain

$$\int_{S.D.} \frac{k_* \exp[-ikx + i(\alpha g k_*)^{1/2}t - k_*(y+a)]\, dk}{2(\alpha g k_*)^{1/2}\{(\alpha g k_*)^{1/2} - kU\}}$$

$$\sim \exp[-s_0(y+a)] \int_{S.D.} \frac{k_* \exp[-ikx + i(\alpha g k_*)^{1/2}t]\, dk}{2(\alpha g k_*)^{1/2}\{(\alpha g k_*)^{1/2} - kU\}}$$

$$= \exp[-s_0(y+a)] \int_{S.D.} g_{II}\, dk, \quad \text{say.} \tag{40}$$

The last integral is equivalent to an integral over the contour Γ_2' (figure 6), provided we again add the appropriate residue term when $s_0 < k_0$ or $x < \frac{1}{2}Ut$; i.e.

$$\int_{S.D.} g_{II}\, dk = \int_{\Gamma_2'} g_{II}\, dk + \frac{2\pi i}{U^2} \exp[-ik_0 x + i(\alpha g k_0)^{1/2}t] H(\tfrac{1}{2}U - \beta). \tag{41}$$

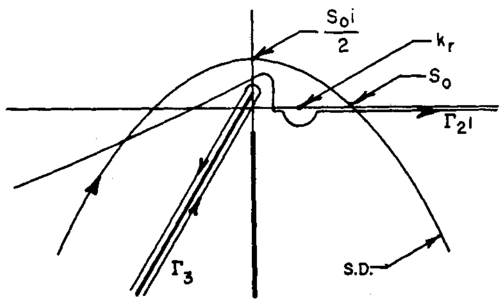

Figure 6. Equivalent inversion path.

The integral over Γ_2' may be evaluated by using formula 809 of Foster & Campbell (1954):

$$\int_{\Gamma_2'} g_{II}\, dk = \frac{1}{2\alpha g} \int_{-\infty}^{\infty} \frac{\exp[-ikx + i(\alpha h k)^{1/2}t]}{1 - U(k/\alpha g)^{1/2}}\, dk$$

$$= \frac{\pi i}{U^2} \exp[-ik_0(x - Ut)] \operatorname{erfc}[(\alpha g/U^2 ix)^{1/2}(\tfrac{1}{2}Ut - x)] +$$

$$+ (\alpha g/i\pi U^2 x)^{1/2} \exp[i(g\alpha t^2/k_0 x)] - \frac{2\pi i}{U^2} \exp[-ik_0(x - Ut)]. \tag{42}$$

Asymptotically, the final result becomes

$$II \sim \frac{2\pi i}{U^2} \exp[-s_0(y+a) - ik_0(x - Ut)]\{\tfrac{1}{2} \operatorname{erfc}[(\alpha g/U^2 ix)^{1/2}(\tfrac{1}{2}Ut - x)] - 1\} +$$

$$+ \frac{2\pi i}{U^2} H(\tfrac{1}{2}U - \beta) \exp[-ik_0(x - Ut)]\{\exp[-s_0(y+a)] - \exp[-k_0(y+a)]\} \tag{43}$$

where $s_0 = \alpha g/4\beta^2$, $k_0 = \alpha g/U^2$, $t = x/\beta$, and

$$\operatorname{erfc} u = 1 - \frac{2}{\sqrt{\pi}}\int_0^u \exp(-v^2)\,dv.$$

For $x \gg \frac{1}{2}Ut$, we have $II \sim 0$; for $0 < x \ll \frac{1}{2}Ut$,

$$II \sim -(2\pi i/U^2)\exp[-k_0(x-Ut)-k_0(y+a)].$$

Thus, for $x < \frac{1}{2}Ut$, II just cancels that contribution of I which arises from the pole at k_r. It may also be shown that $III = II^*$ (complex conjugate). The wave height is therefore

$$\eta = -\frac{\gamma}{2\pi}(I + 2\operatorname{Re} II); \tag{44}$$

and the wave height of the resurgent phenomenon is given asymptotically by

$$\left.\begin{aligned} &\eta_w \sim 0, && x > Ut;\\ &\eta_w \sim \frac{2\pi P_0 a k_0}{\rho g}\exp[-k_0(y+a)]\sin k_0(x-Ut), && \tfrac{1}{2}Ut < x < Ut;\\ &\eta_w \sim 0, && x < \tfrac{1}{2}Ut. \end{aligned}\right\} \tag{45}$$

The other contributions to the solution η do not contribute significantly to the resurgence.

Name	*CAROL*	*EDNA*	—	—
Date	30 Aug.– 1 Sept. 1954	11–12 Sept. 1954	14–15 Sept. 1944	21–22 Sept. 1938
Velocity U (knots)	32–34	32	33	40
Wave Period T (hours)				
comp.: $\alpha = 5{\cdot}0\times10^{-3}$	5·8–6·1	5·8	6·0	7·3
$\alpha = 4{\cdot}2\times10^{-3}$	6·9–7·2	6·9	7·1	8·6
obs.: Atlantic City	5·5	6·0	5·6	—
Sandy Hook	7·0	7·0	7·2	8·0
Duration				
comp.	16–24	17–24	11–12	9
obs.: Atlantic City	20	23	23	—
Sandy Hook	26	—	30	16

Table 1. Periods and durations of sea level disturbance due to four hurricanes.

The resurgent wave motion induced by this simple pressure distribution, equation (25), is essentially restricted to the interval $\frac{1}{2}Ut < x < Ut$, and is characterized by a single wave number $k_0 = \alpha g/U^2$ (that of the fundamental edge wave mode) or the corresponding period $T = 2\pi U/(\alpha g)$. The duration D, which is the time taken for the resurgence to pass a given point, is equal to the total time t for which the disturbance has been in existence and in motion. The periods and durations of the resurgences of several hurricanes have been computed by Munk, Snodgrass & Carrier; the observation points were Atlantic City, where $\alpha = 5{\cdot}0\times10^{-3}$, and Sandy Hook, where $\alpha = 4{\cdot}2\times10^{-3}$. Their results are presented in table 1.

Although the model of equation (25) bears little resemblance to an actual disturbance, the results are in agreement because, evidently, hurricanes also excite only the fundamental edge wave mode (see § 4). The computed periods of the fundamental mode are in excellent agreement with the observed periods.

Mathematically, the existence of a resurgence is due to the fact that the integrand of the Fourier inversion integral has poles at k_r and k_l. The analytic form of the waves is obtained by evaluating the residues of the integrand at these poles. These results may be generalized to the case of an arbitrary pressure distribution which affects all the edge wave modes. The analysis of each integral arising from the inverse Laplace transform of the infinite series, equation (23), is substantially the same as that already performed in equation (26) et seq. The integrals involving the nth Laguerre polynomial give rise to the nth edge-wave mode. The analytic form of the resurgence is again obtained from an evaluation of the residues of the Fourier inversion integral. The final result is that, for disturbances originating at $t = 0$ and travelling parallel to the coast with constant velocity U, the resurgence, which is a sum of all the edge-wave modes, is asymptotically given by

$$\left.\begin{array}{ll} \eta_w \sim 0, & x > Ut; \\ \eta_w \sim \sum_0^\infty \bar{A}_n(k_n)\exp(-k_n y)L_n(2k_n y)\sin k_n(x-Ut), & \tfrac{1}{2}Ut < x < Ut; \\ \eta_w \sim 0, & x < \tfrac{1}{2}Ut, \end{array}\right\} \tag{46}$$

where $\exp(-k_n y)L_n(2k_n y)\sin k_n(x-Ut)$ is the nth mode, its amplitude being $\bar{A}_n(k_n)$, $k_n = (2n+1)\alpha g/U^2 = (2n+1)k_0$. The resurgence is confined to the interval $\frac{1}{2}Ut < x < Ut$; the front moves with the speed of the storm U, and the rear moves with group velocity $\frac{1}{2}U$. The duration is the same as that previously calculated.

We may now apply these results to a more exact mathematical model, no new techniques being needed.

4. The Gaussian pressure distribution

To investigate how the various modes are stimulated by a real disturbance, we need a more realistic model. We consider, therefore, the effects of a travelling pressure taken to have a Gaussian distribution given by

$$P(x,y,t) = P_0 \exp\left[-\frac{(x-Ut)^2+(y-y_0)^2}{a^2}\right], \tag{47}$$

where P_0 is the maximum pressure deviation, and $a = 1{\cdot}19r_0$, r_0 being the half-pressure radius. It is advantageous to use the potential-function formulation of equation (4). The wave height is then given by equation (5). On transforming to a moving coordinate system ($x' = x-Ut$, $y' = y$, $t' = t$) and then dropping the primes, equation (4) becomes

$$y\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)+\frac{\partial\phi}{\partial y}-\frac{1}{\alpha g}\frac{\partial^2\phi}{\partial t^2}+\frac{2U}{\alpha g}\frac{\partial^2\phi}{\partial x^2} = -\frac{U}{\rho g a}\frac{\partial P}{\partial x}. \tag{48}$$

In this coordinate system, the pressure is independent of t. By taking Fourier and Laplace transforms, we obtain

$$y\frac{d^2\Phi}{dy^2}+\frac{d\Phi}{dy}-\left\{yk^2+\frac{(s+iUk)^2}{\alpha g}\right\}\Phi$$
$$=\frac{Uik}{\alpha\rho g s}P_0\exp[-(y-y_0)^2/a^2]\,a\sqrt{\pi}\exp(-a^2k^2/4), \quad (49)$$

where

$$\Phi=\int_{-\infty}^{\infty}e^{ikx}\,dx\int_0^{\infty}e^{-st}\phi\,dt=\int_{-\infty}^{\infty}e^{-st}\bar{\phi}(k,y,t)\,dt.$$

The solution of this equation (we actually solve a modified equation with yk_*^2 in place of yk^2) is, from equation (13),

$$\Phi=-\frac{2\sqrt{\pi}iUP_0\,ak_*}{\rho}\exp(-k_*y)\sum_0^{\infty}\frac{A_n'(k)L_n(2k_*y)}{(n!)^2s\{(s+iUk)^2+(2n+1)k_*\alpha g\}} \quad (50)$$

where

$$A_n'(k)=k\exp(-a^2k^2/4)\int_0^{\infty}L_n(2k_*y)\exp(-k_*y)\exp\{-(y-y_0)^2/a^2\}\,dy. \quad (51)$$

The coefficient $A_n'(k)$ has no poles in the finite k-plane; its branch line and branch points coincide with those of k_*. Taking an inverse Laplace transform, we find that

$$\bar{\phi}(k,y,t)=-\frac{2\sqrt{\pi}iUP_0\,ak_*\exp(-k_*y)}{\rho}\sum_0^{\infty}\frac{A_n'(k)L_n(2ky)}{(n!)^2}\times$$
$$\times\left\{\frac{1}{U^2k^2-(2n+1)\alpha g}+\ldots\right\}. \quad (52)$$

We have written only the first term of three in the brackets, because, as we have mentioned, it may be shown that this term gives the wave motion for $x<Ut$; the other terms add to yield a cancelling wave in the region $x<\frac{1}{2}Ut$. To determine the induced modal amplitudes, we need compute only the wave function arising from the first term, since all additional terms yield the *same* wave.

To compute the amplitudes of the waves constituting the resurgence, we perform the inverse Fourier transform and evaluate the residues arising from the poles, which are at

$$k=\pm k_n=\pm(2n+1)k_0=\pm(2n+1)\alpha g/U^2. \quad (53)$$

In the original coordinates, the result is

$$\left.\begin{aligned}\phi(x,y,t)&=\sum_0^{\infty}\phi_n(x,y,t)+\int F\,dk, && \tfrac{1}{2}Ut<x<Ut;\\ \phi(x,y,t)&=\int F\,dk, && \text{elsewhere},\end{aligned}\right\} \quad (54)$$

where ϕ_n is the wave motion of the nth mode, determined, as we have said, by evaluating the residues of the integrand at the poles $k=\pm k_n$; thus

$$\phi_n(x,y,t)=-\frac{2}{U^2}\left(-\frac{2\sqrt{\pi}UP_0\,a}{\rho}\right)A_n'(k_n)\cos k_n(x-Ut)L_n(2k_ny)\exp(-k_ny). \quad (55)$$

The integral in (54) represents all the other terms, which do not contribute to the resurgence. Since the wave height is related to the potential function by equation (5), the amplitude of the nth edge-wave mode, as defined in equation (45), is

$$\bar{A}_n(k_n) = -\frac{2}{U}\left(-\frac{2\sqrt{\pi}UP_0 a}{\rho}\right)A_n'(k_n)k_n. \tag{56}$$

The amplitude ratio of the nth mode to the mth is

$$\frac{\bar{A}_n}{\bar{A}_m} = \frac{(m!)^2k_n^2\exp(-a^2k_n^2/4)}{(n!)^2k_m^2\exp(-a^2k_m^2/4)} \times \\ \times \frac{\int_0^\infty L_n(2k_n y')\exp(-k_n y')\exp[-(y'-y_0)^2/a^2]\,dy'}{\int_0^\infty L_m(2k_m y')\exp(-k_m y')\exp[-(y'-y_0)^2/a^2]\,dy'}. \tag{57}$$

The amplitude ratios $\bar{A}_1/\bar{A}_0$ and $\bar{A}_2/\bar{A}_0$ are plotted in figures (7) and (8) as functions of $k_0 a$ for $y_0 = 0$ (storm centred on the coast line).

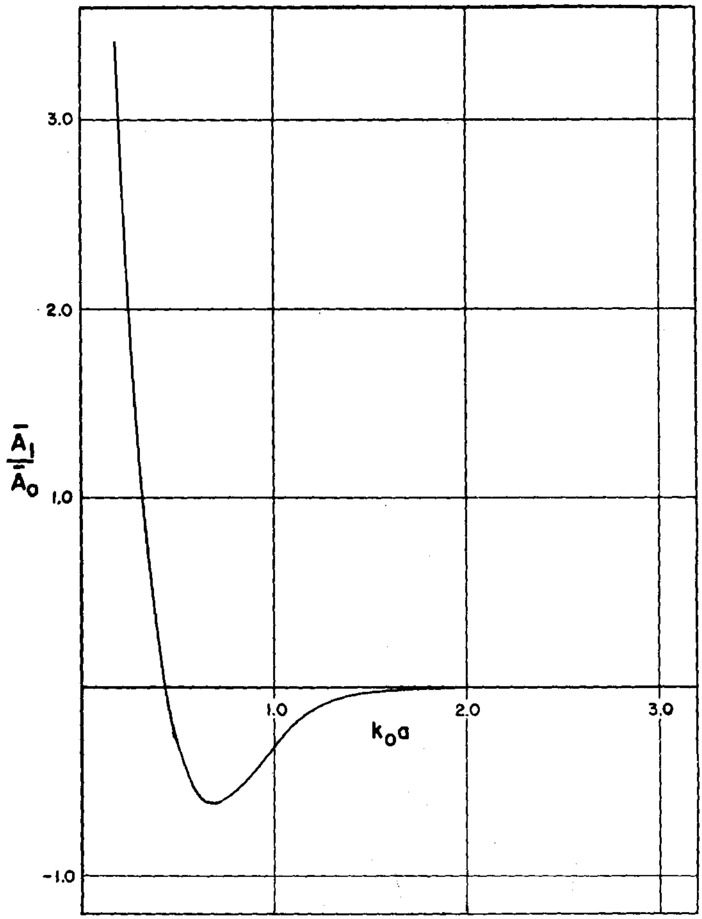

Figure 7. Amplitude ratio of the first harmonic to the fundamental *vs* $k_0 a$.

We shall interpret these results in terms of the half-pressure radius r_0 and the speed of propagation U. For $y = 0$, $P = P_0 \exp(-r^2/a^2)$, or $r = 0{\cdot}837a$. For hurricanes travelling parallel to the east coast of the United States* at 32–34 knots, which is typical, k_0^{-1} lies in the range $30 < k_0^{-1} < 40$ miles. The half-pressure radius of these disturbances is

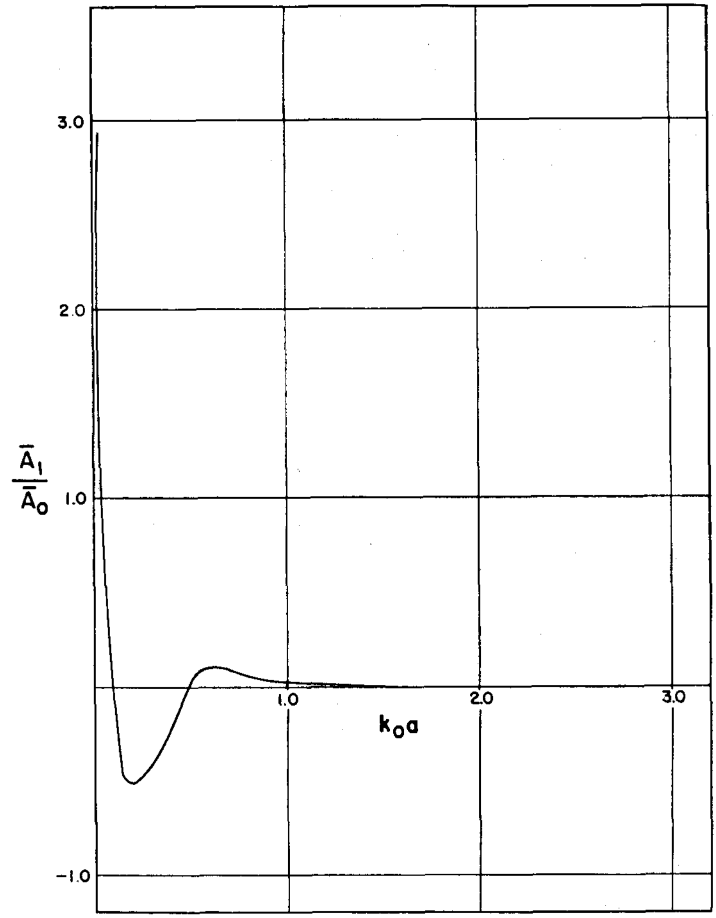

Figure 8. Amplitude ratio of the second harmonic to the fundamental *vs* $k_0 a$.

of the order of 100 miles, which means that a $\doteqdot$ 120 miles, or that $3 < k_0 a < 4$. From figures 7 and 8 it is immediately apparent that for these values of $k_0 a$ only the fundamental edge wave,

$$L_0(2k_0 y) \exp(-k_0 y) \sin k_0(x - Ut),$$

is excited. All other modes are negligibly small. These results also show that for a disturbance of hurricane proportions to be an effective generator

* Henceforth we shall refer exclusively to hurricanes in this location. In analysing the effects of disturbances off other coasts, the appropriate values of α, U, a, etc. must be used.

of the first harmonic, it must travel at 70 knots at least. To excite the second harmonic, it must have speeds of 120 knots or more. We may draw the conclusion that hurricanes will never effectively stimulate any other mode except the fundamental edge wave. For smaller disturbances the higher modes may be generated. A storm with a half-pressure radius of approximately 25 miles and a speed of 34 knots has a value of $k_0 a$ between 0·75 and 1; k_0 is thus located exactly in the region where higher harmonics are stimulated. To induce still higher harmonics, disturbances must travel faster or be smaller in size or both.

Let us investigate the fundamental mode, and determine the conditions (size, distance from shore, etc.) for optimum edge-wave generation. The wave form of the fundamental mode is given by

$$\eta_0 = \left[\frac{2\pi P_0}{\rho g}(k_0 a)^2 \exp(-k_0 a)\operatorname{erfc}\left(-\frac{y_0}{a}+\frac{k_0 a}{2}\right)\right]\exp(-k_0 y)\sin k_0(x-Ut). \tag{58}$$

The bracketed expression is the wave amplitude. To find the storm position y_0 that produces optimum amplitudes (with constant $k_0 a$), we must determine the value of y_0, say $y_0^{\max}$, which maximizes

$$f(y_0) = \exp(-k_0 y_0)\operatorname{erfc}\left(-\frac{y_0}{a}+\frac{k_0 a}{2}\right),$$

that is, for which $f'(y_0) = 0$. Hence, we need to solve

$$\operatorname{erfc} u = \frac{2}{\sqrt{\pi}}\frac{\exp(-u^2)}{k_0 a}, \quad \text{where } u = -y_0/a+\tfrac{1}{2}k_0 a.$$

For $k_0 a = 2$, 2·5, 3·0, 3·5, we find $u \doteqdot 0·6$, 0·9, 1·2, 1·5. In general these results yield maximum wave heights for y_0 in the range 25 to 35 miles. Table 2 summarizes these results, giving the maximum wave height in terms of the coefficient $2\pi P_0/(\rho g)$. For a maximum pressure difference of one pound per square inch, the resulting wave heights, in the hurricane region $k_0 a = 3·5$, are approximately 2·5 ft. It is seen that the wave height measured at the coast varies slightly with the distance of the storm centre from shore, being 2·3 ft. for $y_0 = 0$, and rising to a maximum of 2·5 ft.

Knowing that maximum wave heights are induced by hurricanes 24 to 35 miles from shore, we can now determine what size disturbance in this region will produce the largest effects. We maximize the wave amplitude with respect to the variable a, holding k_0 and y_0 fixed. This leads to the equation

$$\sqrt{\pi}\operatorname{erfc}(-y_0/a+\tfrac{1}{2}k_0 a) = (y_0/a+\tfrac{1}{2}k_0 a)\exp[-(y_0/a-\tfrac{1}{2}k_0 a)^2].$$

For $k_0 = 0·025$ and $y_0 = 30$, $a \doteqdot 68$ miles, and the half-pressure radius is $r_0 \doteqdot 54$ miles. Our conclusion is that hurricanes are too big to produce maximum wave heights; 'smaller' storms ($r_0 \doteqdot 50$ miles) 25 to 35 miles from shore produce the largest waves (for a given pressure deviation P_0).

$k_0\, a$	Maximum amplitude	k_0 and a	Storm centre (y_0^{max})
2·0	$0{\cdot}712\ \dfrac{2\pi P_0}{\rho g}$	1/30 & 60 1/40 & 80	22·8 30·4
2·5	0·529 (,,)	1/30 & 75 1/40 & 100	25·0 34·0
3·0	0·328 (,,)	1/30 & 90 1/40 & 120	27·0 36·0
3·5	0·170 (,,)	1/30 & 105 1/40 & 142	26·2 37·0

In particular, if $P_0 = 1$ lb/in.^{2}, and $\rho g = 62{\cdot}5$ lb/ft.^{3}, then $2\pi P_0/(\rho g) = 14{\cdot}5$, and the following wave heights are computed for various distances y_0.

$k_0\, a$	$y_0 = 0$	$y_0 = y_0^{max}$	$y_0 = 100$
2·0	—	10·4	—
2·5	—	7·65	—
3·0	4·44	4·75	2·7–3·7
3·5	2·3	2·46	1·65–2·02

Table 2. Maximum wave heights in fundamental mode : a and y_0 are in miles, and wave heights in feet.

5. Conclusion

We can now predict the effects of any particular distribution on the resurgence from a knowledge of the maximum pressure difference, half-pressure radius, and speed. One should hence be able to calculate the positions of highest tide along the east coast following hurricanes. While the amplitude of the resurgence is not in itself remarkable, the extremely long periods enable these waves to be effectively superimposed upon an incoming tide; this is a dangerous situation, especially in those areas where there are resonant phenomena. A two foot or three foot crest coming several hours after the passage of a storm (and therefore somewhat unexpectedly), and superimposed upon the incoming tide, may greatly enlarge existing resonant wave motions (in bays, estuaries, etc.). While these situations cannot be avoided, it may be possible to reduce their effects by predicting their occurrence.

It should be noted that in order to simplify the mathematical analysis the effect of the earth's rotation has been neglected. This approximation is quite proper for the study of wave motion whose period is a few hours or less. However in the application of the theory to hurricane resurgences it is estimated that errors of 15 % are probable. That we were able to obtain such excellent and consistent agreement between observation and theory with so glaring an omission is, to say the least, peculiar. At present those additional effects which presumably nullify the rather large contribution from rotation are unknown.

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